Computer graphics III – Multiple Importance Sampling

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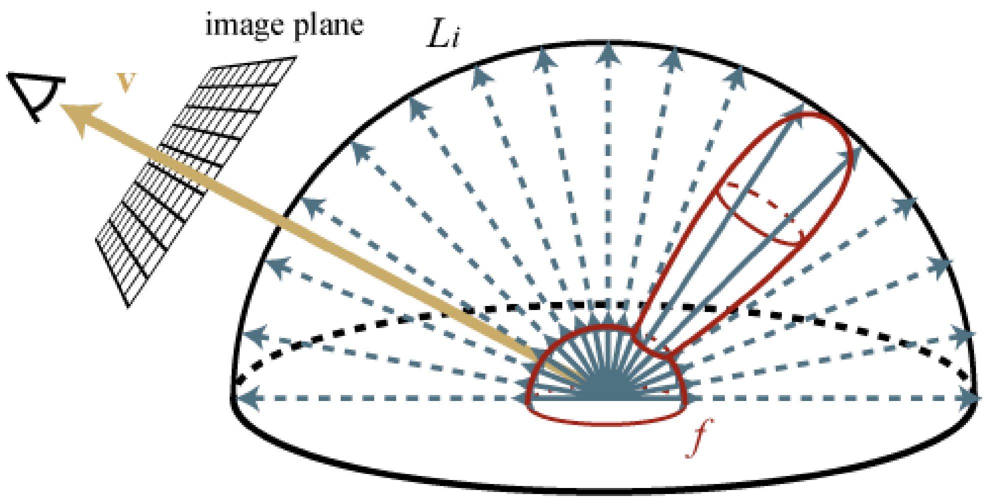
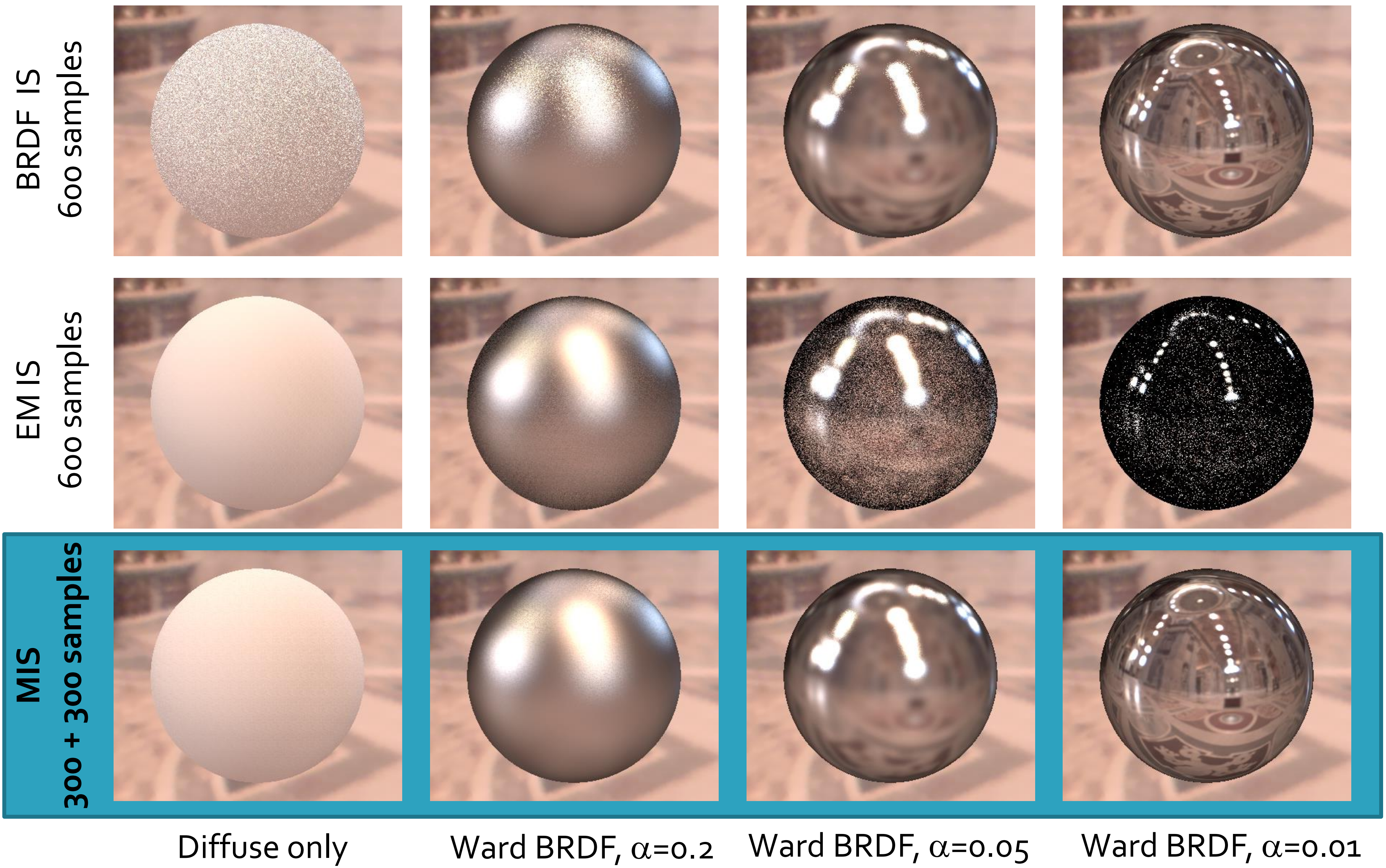
Sampling of environment lighting



$$L_{\text{out}}(\omega_{\text{out}}) = \int_{H(\mathbf{x})} L_{\text{in}}(\omega_{\text{in}}) \cdot f_r(\omega_{\text{in}} \to \omega_{\text{out}}) \cdot \cos\theta_{\text{in}} \; \mathrm{d}\omega_{\text{in}}$$

Sampling of environment lighting

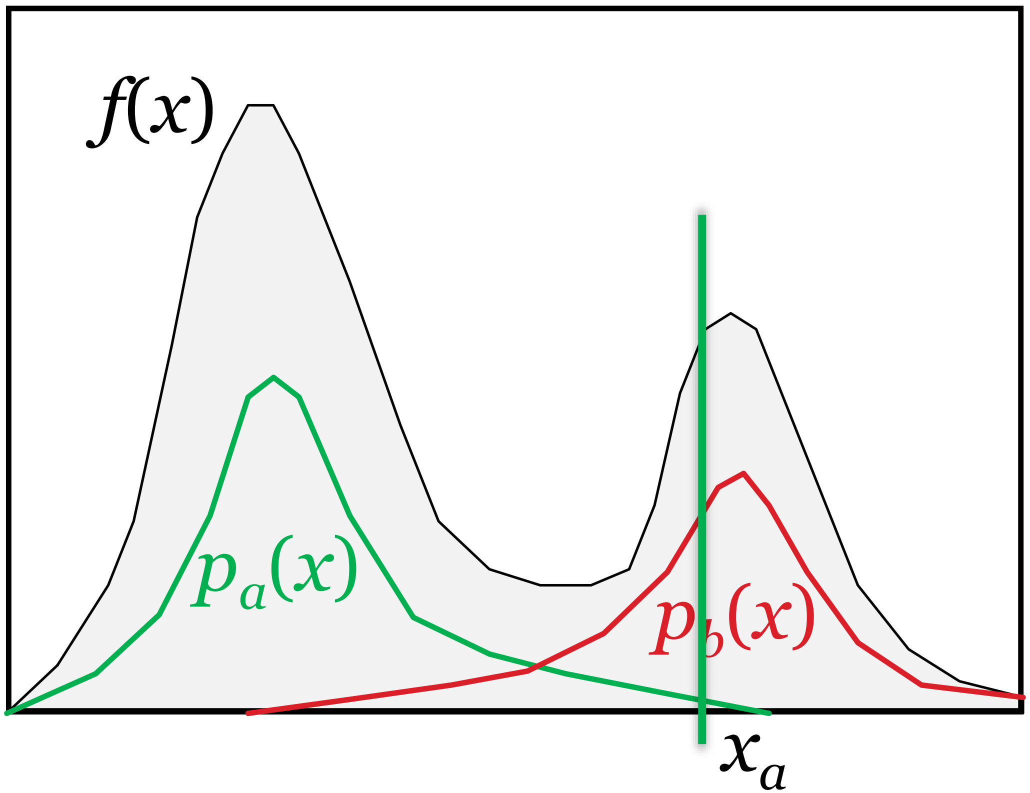


Sampling of environment lighting

- Two different sampling strategies for generating the incoming light direction ω_{in}

1. **BRDF-proportional sampling -** $p_a(\omega_{in})$
2. **Environment map-proportional sampling -** $p_b(\omega_{in})$

What is wrong with using either of the two strategies alone?



Notes on the previous slide

- We have a complex multimodal integrand g(x) that we want to numerically integrate using a MC method with importance sampling. Unfortunately, we do not have a PDF that would mimic the integrand in the entire domain. Instead, we can draw the sample from two different PDFs, p_a and p_b each of which is a good match for the integrand under different conditions – i.e. in different part of the domain.
- However, the estimators corresponding to these two PDFs have extremely high variance – shown on the slide. We can use Multiple Importance Sampling (MIS) to combine the sampling techniques corresponding to the two PDFs into a single, robust, combined technique. The MIS procedure is extremely simple: sample from both techniques p_a and p_b , and then weight the samples appropriately.

- This estimator is really powerful at suppressing outlier samples such as those that you would obtain by picking x from the tail of p_a, where g(x) might still be large. Without having p_b at our disposal, the MC estimator would be dividing the large g(x) by the small p_a (x), producing an outlier sample.
- The combined technique has a much higher chance of producing this particular x (because it can sample it also from p_b), so the combined estimator divides g(x) by [p_a (x) + p_b(x)] / 2, which yields a much more reasonable sample value.

- I want to note that what I'm showing here is called the "balance heuristic" and is a part of a wider theory on weighted combinations of estimators proposed by Veach and Guibas.

Multiple Importance Sampling

Multiple Importance Sampling

- Given n sampling techniques (i.e. pdfs) $p_1(x), .. , p_n(x)$
- We take n_i samples $X_{i,1}, .. , X_{i,n_i}$ from each technique
- **Combined estimator**

Combination weights
(different for each sample)

$$F = \sum_{i=1}^{n} \frac{1}{n_i} \sum_{j=1}^{n_i} w_i(X_{i,j}) \frac{f(X_{i,j})}{p_i(X_{i,j})}$$

sampling techniques

samples from individual techniques

Unbiasedness of the combined estimator

- The MIS estimator is unbiased…

$$E[F] = \quad \ldots = \int \left[\sum_{i=1}^{n} w_i(x)\right] f(x)\ \mathrm{d}x \equiv \int f(x)$$

- … provided the weighting functions sum up to 1

$$\forall x: \qquad \sum_{i=1}^{n} w_i(x) = 1$$

Choice of the weighting functions

- **Objective:** minimize the variance of the combined estimator

1. Arithmetic average (very bad combination)

$$w_i(x) = \frac{1}{n}$$

2. **Balance heuristic** (very good combination)
   - ....

Balance heuristic

- Combination weights

$$\hat{w}_i(\mathbf{x}) = \frac{n_i p_i(\mathbf{x})}{\sum_k n_k p_k(\mathbf{x})}$$

- Resulting estimator (after plugging the weights)

$$F = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \frac{f(X_{i,j})}{\sum_k n_k p_k(X_{i,j})},$$

  - The contribution of a sample does not depend on which technique (pdf) it came from
  - Effectively, the sample is drawn from a weighted average of the individual pdfs – as can be seen from the form of the estimator

MIS estimator with the Balance heuristic

- Plugging Balance heuristic weights into the MIS formula

$$F = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \frac{f(X_{i,j})}{\sum_k n_k p_k(X_{i,j})},$$

  - The contribution of a sample does not depend on which technique (pdf) it came from
  - Effectively, the sample is drawn from a weighted average of the individual pdfs – as can be seen from the form of the estimator

Balance heuristic

- The balance heuristic **is almost optimal** [Veach 97]
  - No other weighting has variance much lower than the balance heuristic

- Our work [Kondapaneni et al. 2018] revises MIS
  - If you allow negative weights, one can improve over the balance heuristic a lot

MIS for direct illumination from enviro lights

Application of MIS to environment light sampling

- Recall: Two sampling strategies for generating the incident direction ω_i

1. **BRDF-proportional sampling -** $p_a(\omega_{in})$
2. **Environment map-proportional sampling -** $p_b(\omega_{in})$

- Plug formulas for $p_a(\omega_{in})$ and $p_b(\omega_{in})$ into the general MIS formulas above

Direct illumination: Two strategies

- Which strategy should we choose?
  - **Both!**
- Both strategies estimate the same quantity $L_{\text{out}}(\mathbf{x}, \omega_{\text{out}})$
  - A mere sum would estimate $2 \times L_{\text{out}}(\mathbf{x}, \omega_{\text{out}})$, which is wrong
- We need a weighted average of the techniques, but **how to choose the weights**? → MIS

MIS weight calculation

MIS weight for a sample direction generated by BRDF lobe sampling

$$w_a(\omega_{\text{in},j}) = \frac{p_a(\omega_{\text{in},j})}{p_a(\omega_{\text{in},j}) + p_b(\omega_{\text{in},j})}$$

PDF for BRDF sampling

PDF with which the direction $\omega_{\text{in},j}$ would have been generated, if we used env map sampling

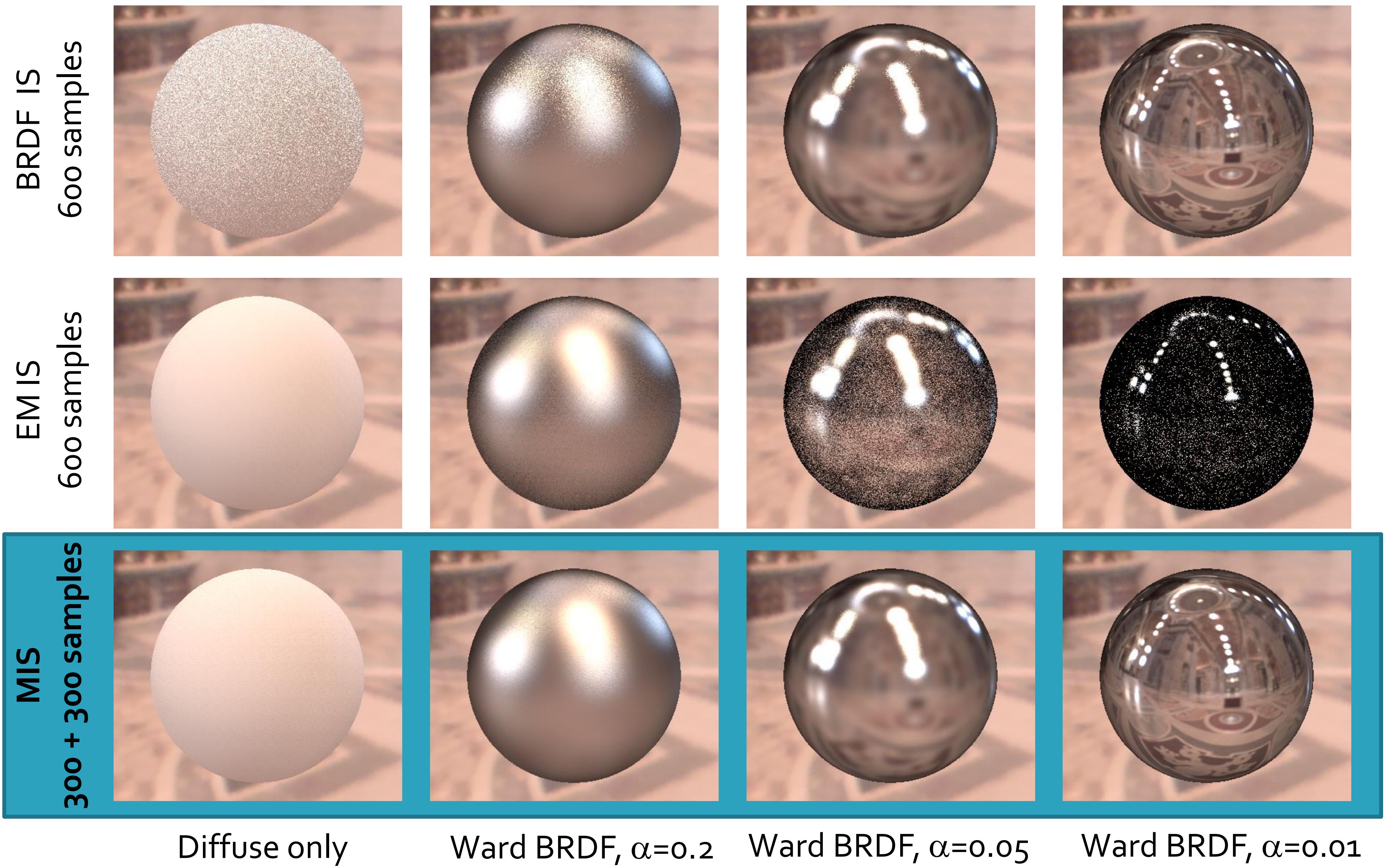
- Here, we assume one sample from each of the two strategies

MIS for enviro sampling – Algorithm

```
Vec3 omegaInA = generateBrdfSample();
float pdfA = evalBrdfPdf(omegaInA);
float pdfAsIfFromB = evalEnvMapPdf(omegaInA);
float misWeightA = pdfA / (pdfA + pdfAsIfFromB);
Rgb outRadianceEstimate = misWeightA *
        incRadiance(omegaInA) *
        brdf(omegaOut, omegaInA) *
        max(0, dot(omegaInA, surfNormal);

Vec3 omegaInB = generateEnvMapSample();
float pdfB = evalEnvMapPdf(omegaInB);
float pdfAsIfFromA = evalBrdfPdf(omegaInB);
float misWeightB = pdfB / (pdfB + pdfAsIfFromA);
outRadianceEstimate += misWeightB *
        incRadiance(omegaInB) *
        brdf(omegaOut, omegaInB) *
        max(0, dot(omegaInB, surfNormal);
```

MIS applied to enviro sampling



MIS for direct illumination from area lights

Area light sampling – Motivation

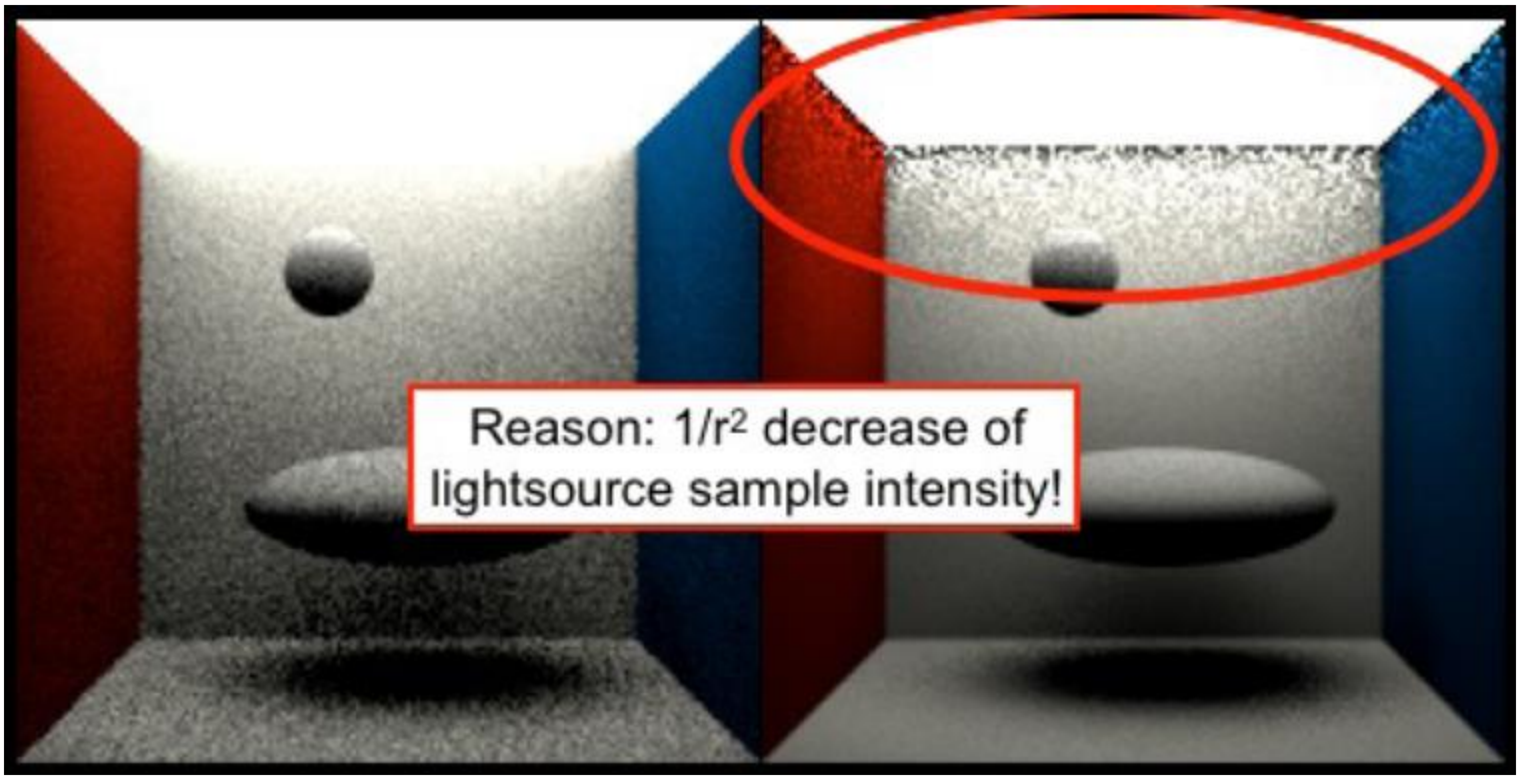


Image: Alexander Wilkie

Sampling technique (pdf) p_a: BRDF sampling

Sampling technique (pdf) p_b: Light source area sampling

MIS-based combination

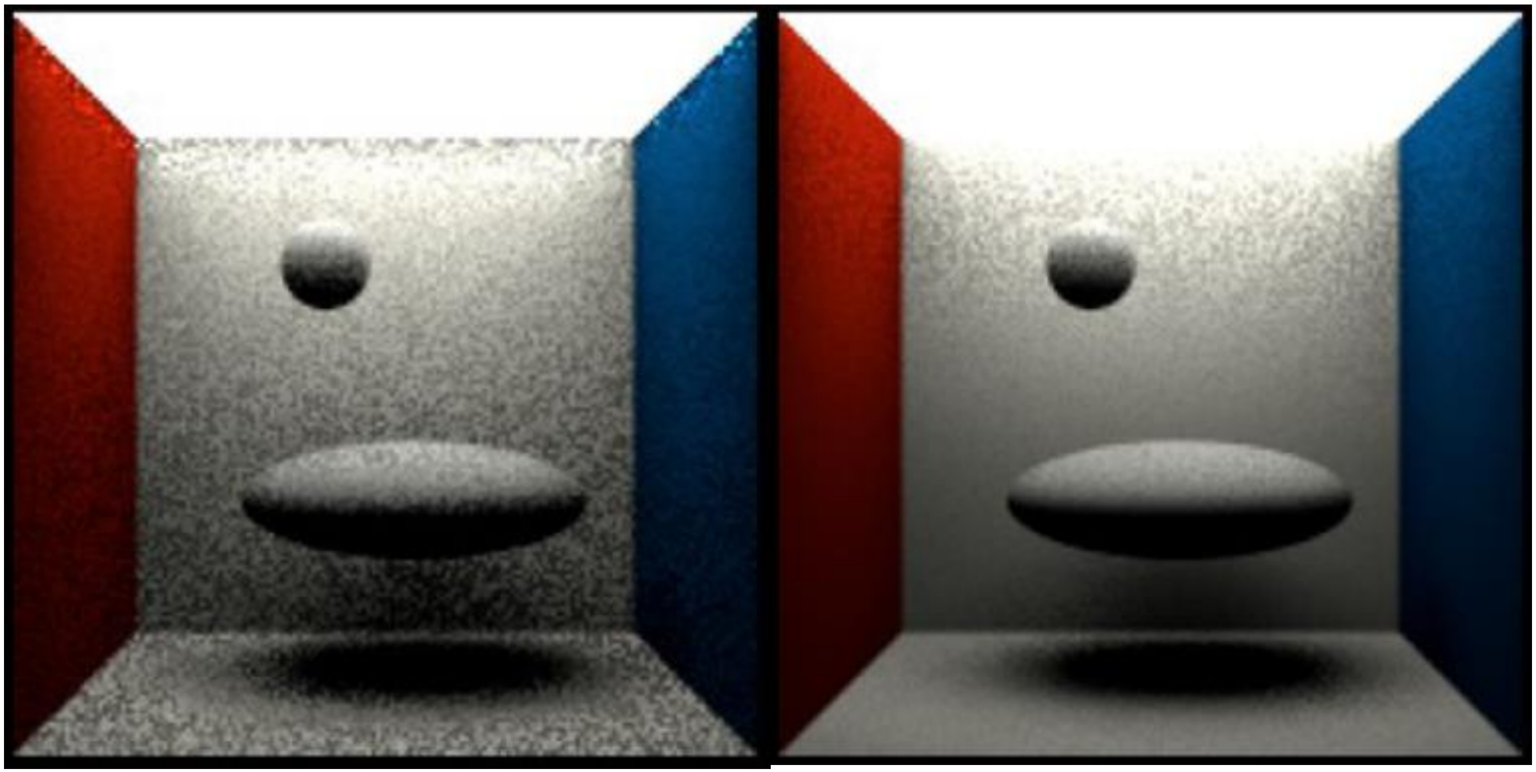
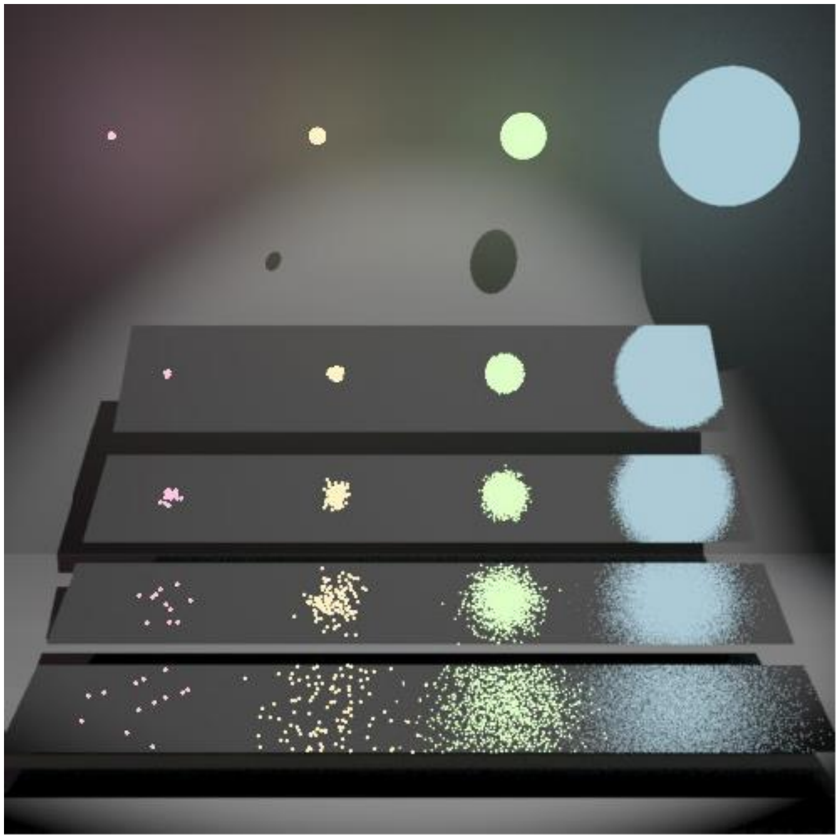


Image: Alexander Wilkie

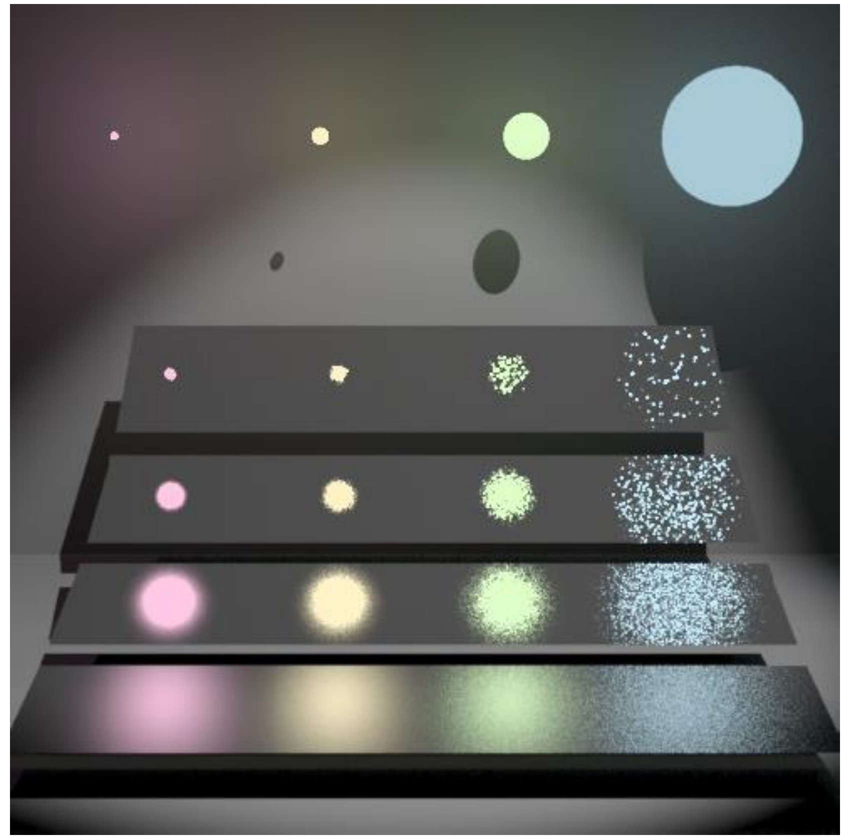
Arithmetic average
Preserves **bad** properties
of both techniques

MIS w/ the balance heuristic
Bingo!!!

Area light sampling – Classic Veach's example



BRDF proportional sampling



Light source area sampling

Images: Eric Veach

MIS-based combination

- **Multiple importance sampling** & **Balance heuristic** (Veach & Guibas, 95)

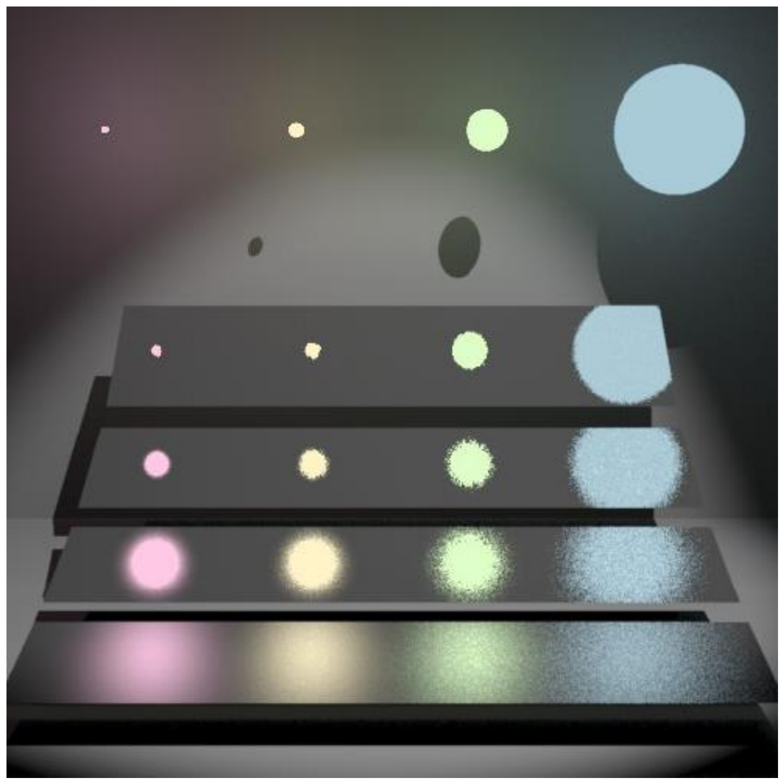


Image: Eric Veach

Direct illumination: Two strategies

- **BRDF proportional sampling**
  - Better for large light sources and/or highly glossy BRDFs
  - The probability of hitting a small light source is small -> high variance, noise
- **Light source area sampling**
  - Better for smaller light sources
  - It is the only possible strategy for point sources
  - For large sources, many samples are generated outside the BRDF lobe -> high variance, noise

Example PDFs

- **BRDF sampling: $p_a(\omega)$**
  - Depends on the BRDF, e.g. the formulas for physically-based Phong BRDF from the last lecture
- **Light source area sampling: $p_b(\omega)$**

$$p_b(\omega) = \frac{1}{|A|} \frac{||\mathbf{x} - \mathbf{y}||^2}{\cos \theta_{\mathbf{y}}}$$

Conversion of the uniform pdf 1/|A| from the area measure (dA) to the solid angle measure (dω)

Contributions of the sampling techniques

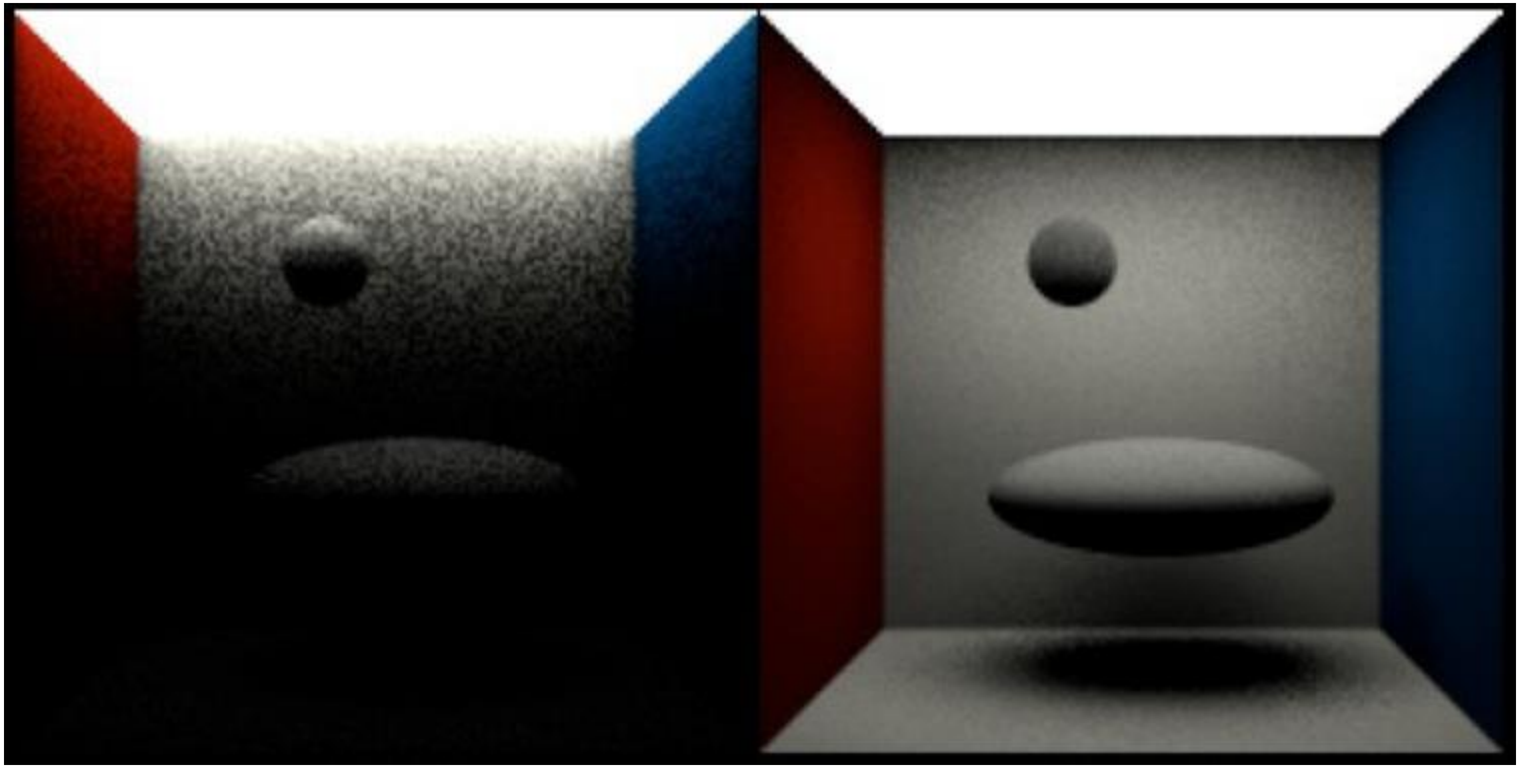


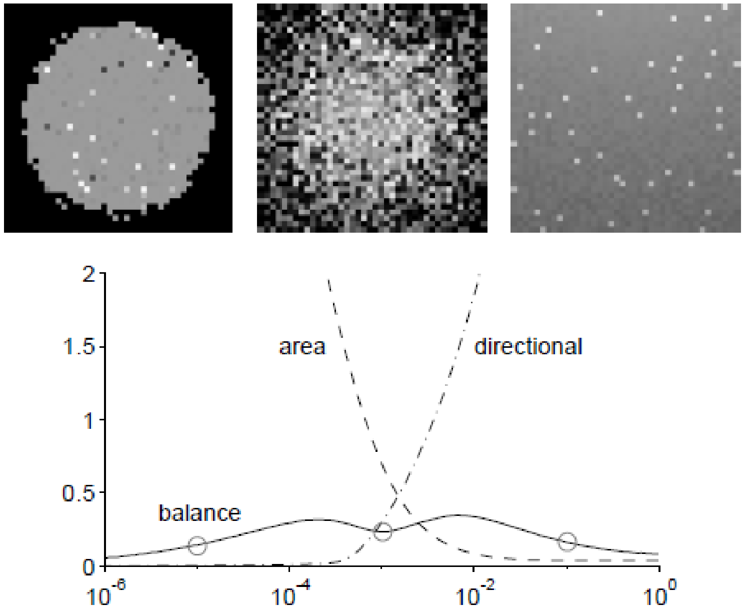
Image: Alexander Wilkie

w_a * BRDF sampling **w_b * light source area sampling**

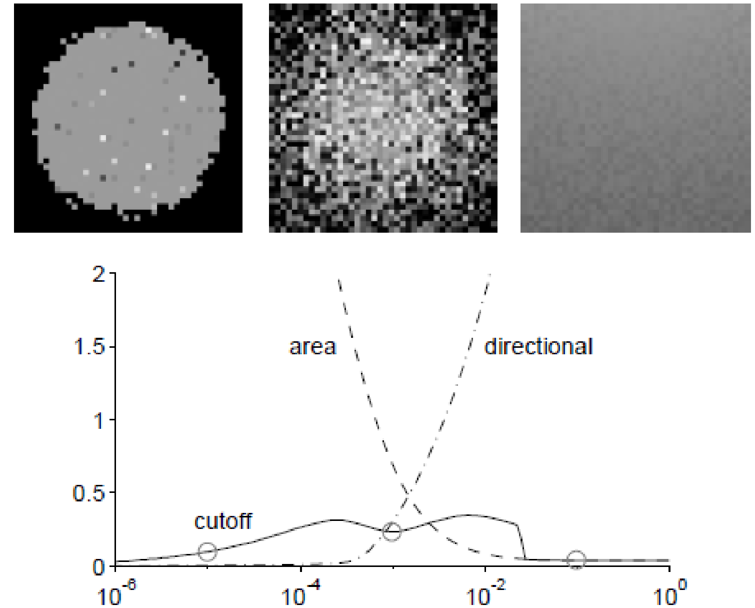
Alternative MIS heuristics

Alternative combination heuristics

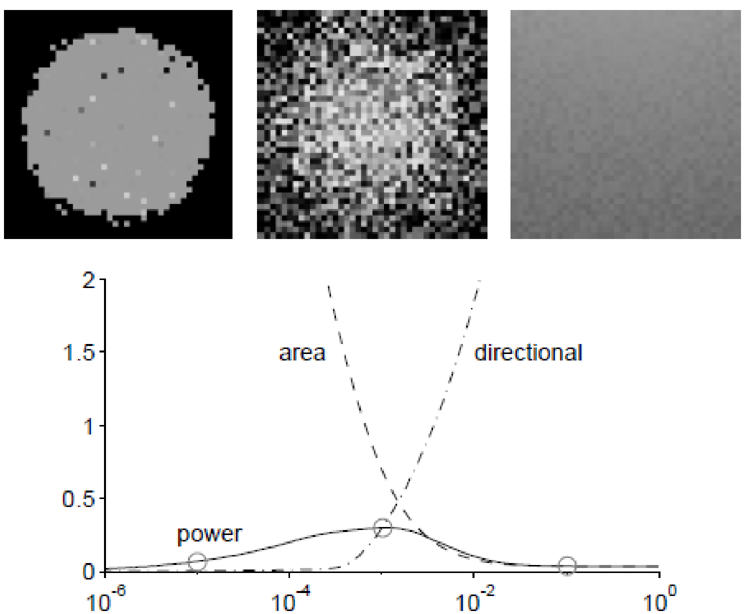
- **"Low variance problems"**
- Whenever one sampling technique yields a very low variance estimator, balance heuristic can be suboptimal
- "Power heuristic" or other heuristics can be better in such a case – see next slide



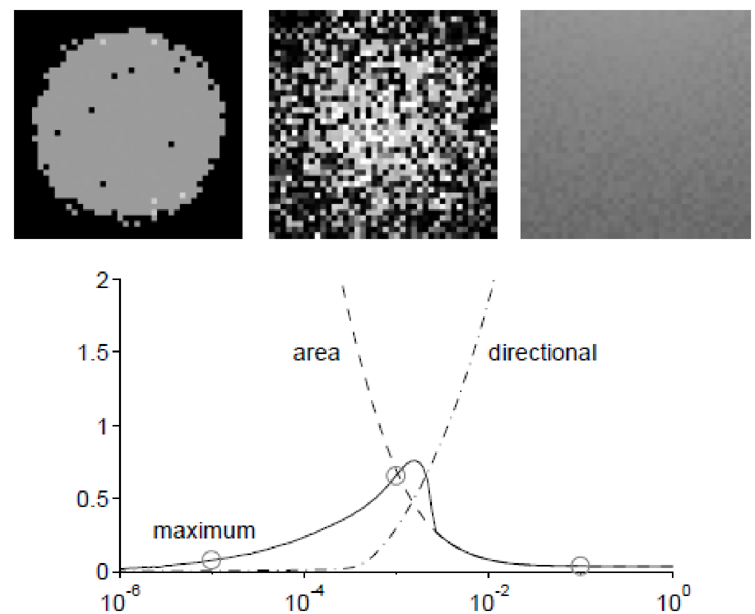
(a) The balance heuristic.



(b) The cutoff heuristic ($\alpha = 0.1$).



(c) The power heuristic ($\beta = 2$).

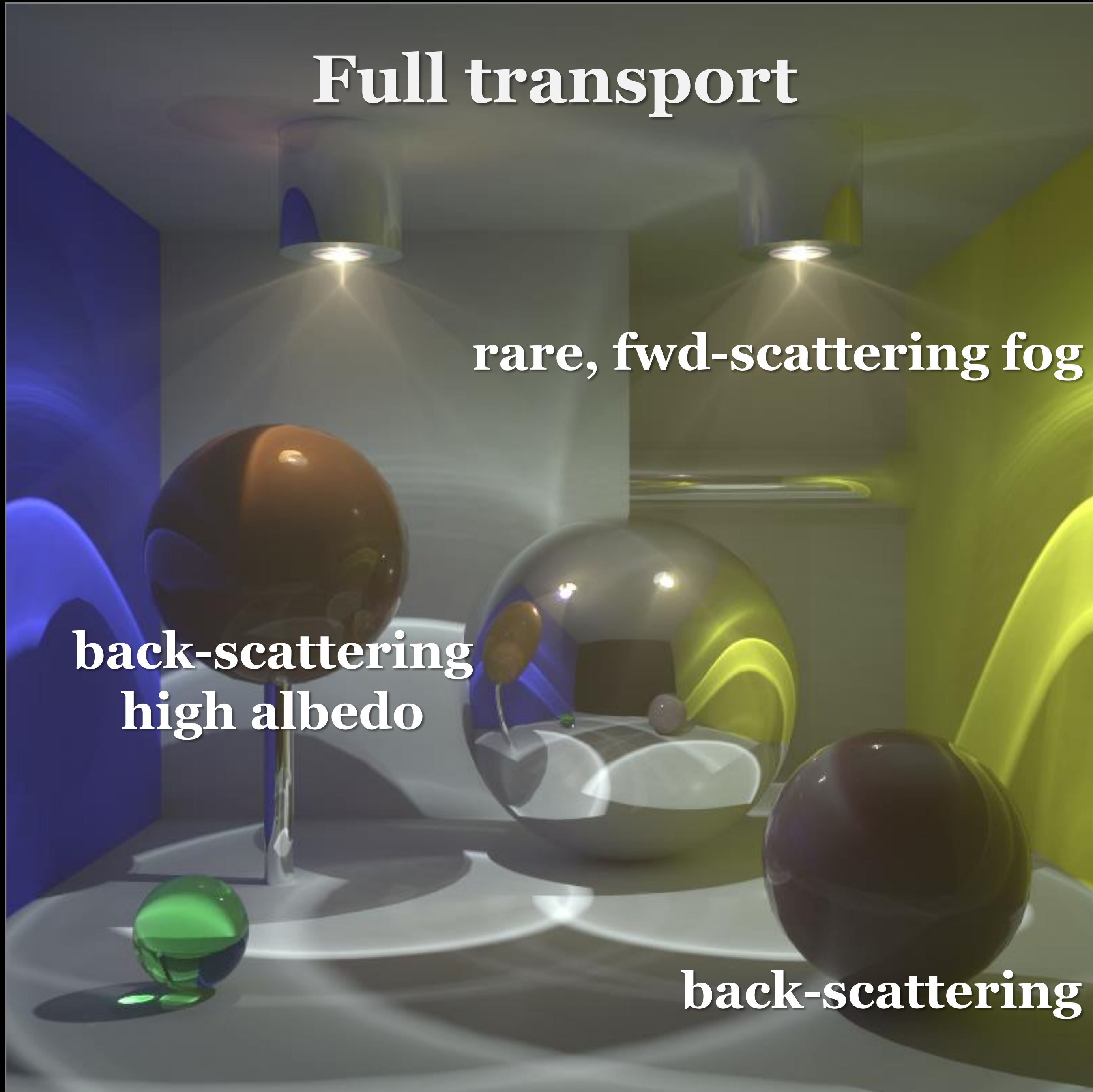


(d) The maximum heuristic.

Other examples of MIS applications

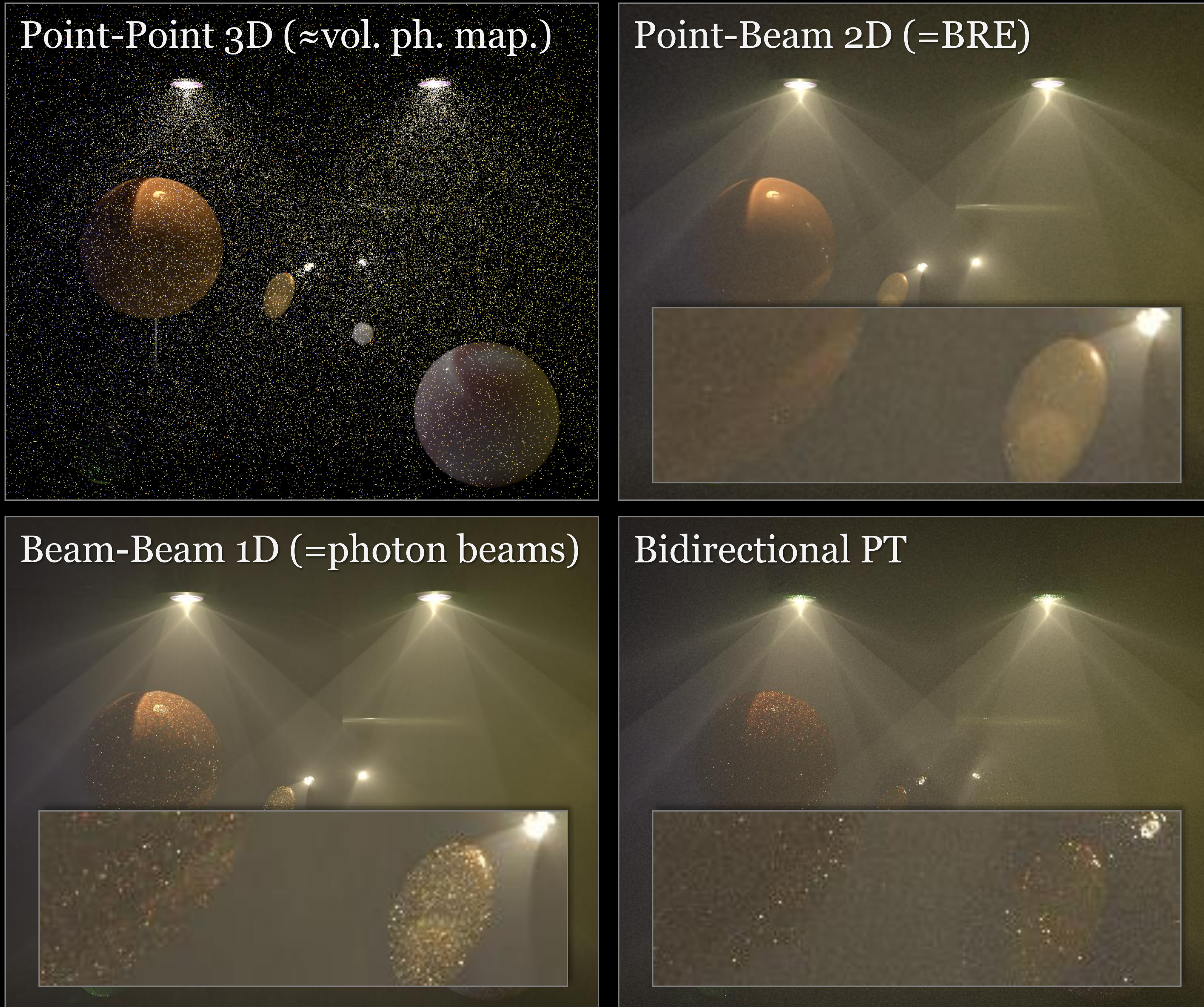
In the following we apply MIS to combine full path sampling techniques for calculating light transport in participating media.

Full transport

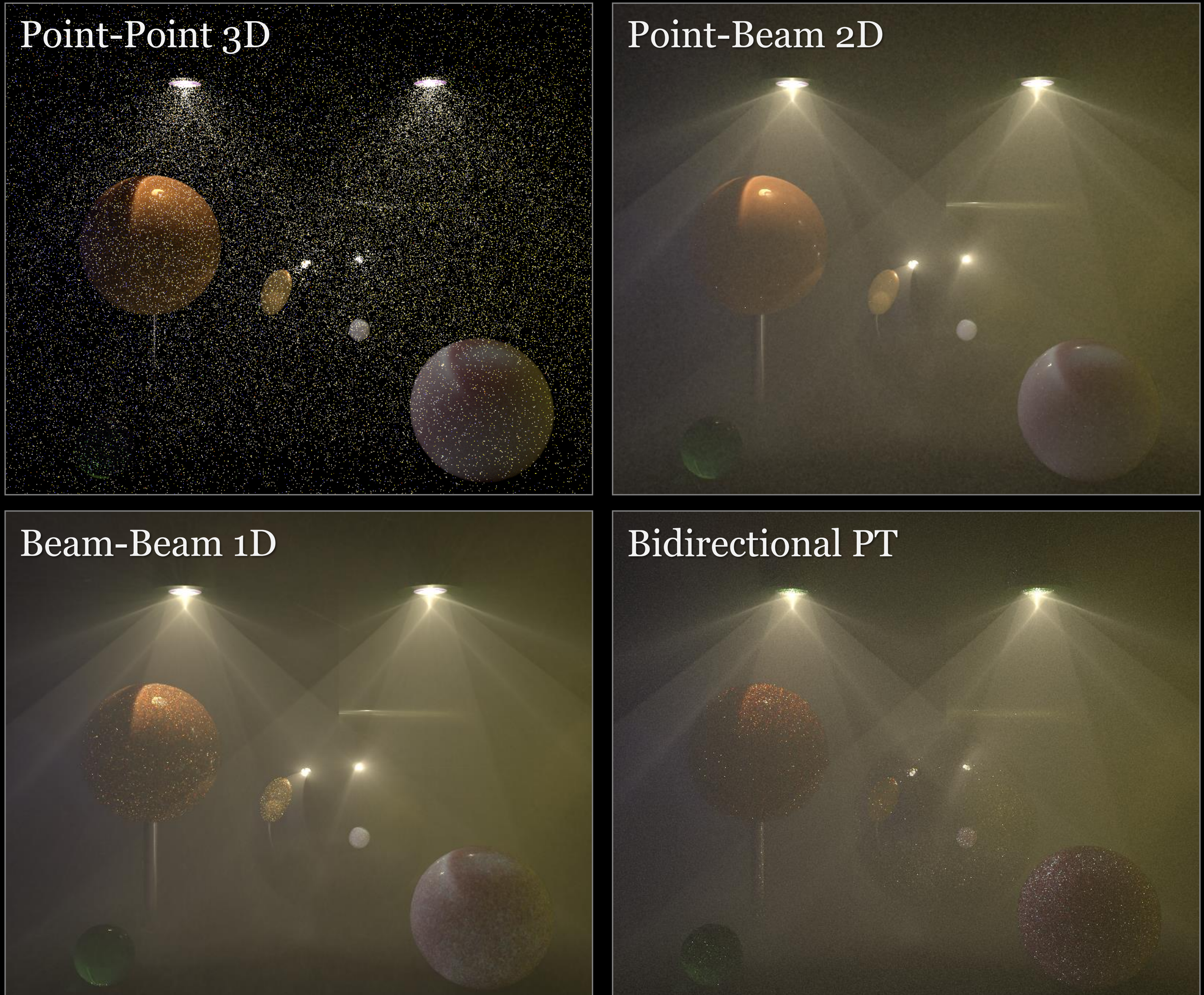


Medium transport only

Previous work comparison, 1 hr



Previous work comparison, 1 hr



Weighted contributions

UPBP (our algorithm)
1 hour